Chapter 45
A Best Theory Diagram for Metallic and Laminated Shells

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Abstract In this work, refinements of classical theories are proposed in order to analyze isotropic, orthotropic and laminated plates and shells. Higher order theories have been implemented according to the Carrera Unified Formulation (CUF) and, for a given problem, the effectiveness of each employed generalized displacement variable has been established, varying the thickness ratio, the orthotropic ratio and the stacking sequence of the lay-out. A number of theories have therefore been constructed imposing a given error with respect to the available 'best solution'. The results have been restricted to the problems for which closed-form solutions are available. These show that the terms that have to be used according to a given error vary from problem to problem, but they also vary when the variable that has to be evaluated (displacement, stress components) is changed.

Keywords Refined classical theories · Laminated shells and plates · Unified formulation

45.1 Introduction

Laminated structures such as traditional composite panels are frequently found in aerospace vehicle applications. High transverse shear and normal deformability as

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well as discontinuity of physical properties make the use of structural models difficult. Accurate stress and strain fields evaluation demand the development of ad hoc theories for the analysis of these structures.

Most known theories were originated from the intuition of some structural analysis pioneers. Among these, Kirchhoff [25], Love [29], Reissner [38], Mindlin [30], Vlasov [42], Koiter [26] and Naghdi [31], and others. In most cases, these 'axiomatic' intuitions lead to a simplified kinematics of the true three-dimensional deformation state of the considered structure: the section remains plane, the thickness deformation can be discarded, shear strains are negligible, etc... For a complete review of this topic, including laminated composite structures, the readers can refer to the many available survey articles on plates and shells [28]- [36].

As an alternative to the axiomatic approach, approximated theories have been introduced employing 'asymptotic-type' expansions of unknown variables over the thickness. The order of magnitude of significant terms is evaluated referring to a geometrical parameter (thickness-to-length in the case of plates and shells). The asymptotic approach furnishes consistent approximations. This means that all the retained terms are those which have the same order of magnitude as the introduced perturbation parameter when the latter vanishes. Articles on the application of asymptotic methods to shell structures can be found in Cicala [17], Fettahlioglu and Steele [19], Berdichevsky [1, 2], Widera and coauthors [43, 44], and Spencer et alii [40], as well as in the monographes by Cicala [18] and Gol'denweizer [20].

Both the axiomatic and the asymptotic methods have historically been motivated by the need to work with simplified theories that are capable of leading to simple formulas and equations which can be solved by hand calculation. Up to five decades ago, in fact, it was quite prohibitive to solve problems with many unknowns (more than 5, 6); nowadays, this limitation no longer holds. Of course, the formulation of more complicated problems would be difficult without the introduction of appropriate techniques that are suitable for computer implementations. The approach herein discussed makes use of such a suitable condensed notation technique that was introduced by the first author during the last decade and it is referred to as the Carrera Unified Formulation, CUF, for beams, plates and shell structures [3,4,6–8,10]. Governing equations are given in terms of a few 'fundamental nuclei' whose form does not depend on either the order of the introduced approximations or on the choices made for the base functions in the thickness direction.

In short, CUF makes it possible to implement those terms which had been neglected by the above cited pioneers. In order to obtain more general conclusions and to draw general guidelines and recommendations in building bidimensional theories for metallic and composite plates and shells, it would be of great interest to evaluate the effectiveness of each refined theory term. This has been done in the present paper. In CUF, in fact, the role of each displacement variable in the solution is investigated by measuring the loss of accuracy due to its being neglected. A term is considered ineffective, i.e. negligible, if it does not affect the accuracy of the solution with respect to a reference 3D solution. Reduced kinematics models, based on a set of retained displacement variables, are then obtained for each considered configuration. Full and reduced models are then compared in order to highlight the sensitivity of a kinematics model to variations in the structural problem. This method can somehow be considered as a *mixed axiomatic/asymptotic approach* since it furnishes asymptotic-like results, starting from a preliminary axiomatic choice of the base functions. A companion investigation, related to plates, has been proposed in [14, 15].

The present work deals with problems with only displacement variables formulated using the Principle of Virtual Displacements (PVD). The results have been restricted to simply supported orthotropic plates and shells, subjected to harmonic distributions of transverse pressure for which closed-form solutions are available. Among the theories contained in the CUF, only the Equivalent Single Layer (ESL) models are considered, in which a laminate plate/shell is reduced to a single lamina of equivalent characteristics. The effectiveness of each displacement variable has been established, varying the thickness ratio, the orthotropic ratio and the stacking sequence of the lay-out. For a given problem, the best theory has been constructed imposing a given error with respect to the available best results.

45.2 Carrera Unified Formulation

The main feature of the Carrera Unified Formulation (CUF) is the unified manner in which a large variety of beam/plate/shell structures are modeled. Details of CUF can be found in the already mentioned papers [3, 4, 6, 7, 10]. According to this formulation, the governing equations are written in terms of a few *fundamental nuclei* which do not formally depend on both the order of the expansion N and the approximating functions, in the thickness direction. In the case of ESL approach, which is herein considered, the displacement field is modeled in the following manner:

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \qquad \tau = 1,2,....,N \tag{45.1}$$

where F_τ are functions of z. $\mathbf{u}_\tau$ is the displacements vector and N stands for the order of the expansion. According to Einstein's notation, the repeated subscript τ indicates summation. In this work, Taylor polynomials are used for the expansion:

$$F_\tau = z^{\tau-1}, \qquad \tau = 1,2,....,N \tag{45.2}$$

N is assumed to be as high as 4. Therefore the displacement field is:

$$\begin{aligned} u &= u_1 + z\,u_2 + z^2\,u_3 + z^3\,u_4 + z^4\,u_5 \\ v &= v_1 + z\,v_2 + z^2\,v_3 + z^3\,v_4 + z^4\,v_5 \\ w &= w_1 + z\,w_2 + z^2\,w_3 + z^3\,w_4 + z^4\,w_5 \end{aligned} \tag{45.3}$$

The Reissner-Mindlin model [30, 38] (also known as First Order Shear Deformation Theory, FSDT, in the case of laminates) for the plate and the Naghdi model [31] for the shell, can be obtained acting on the F_τ expansion. Two conditions have to be imposed. 1) First-order approximation kinematics field, 2) the displacement component w has to be constant above the cross-section, i.e. $w_2 = 0$. The resultant displacement model is:

$$\begin{aligned} u &= u_1 + z\, u_2 \\ v &= v_1 + z\, v_2 \\ w &= w_1 \end{aligned} \tag{45.4}$$

The Kirchhoff-type approximation [25] (also known as Classical Laminate Theory, CLT) for plate and the Koiter approximation [26] for the shell, can also be obtained using a penalty technique for the shear correction factor.

45.2.1 Governing Differential Equations

In this work, the Principle of Virtual Displacements (PVD) is used to obtain the governing equations and boundary conditions. In the general case of multi-layered plates/shells subjected to mechanical loads, the governing equations are:

$$\delta \mathbf{u}_s^{k^T} : \qquad \mathbf{K}_{uu}^{k\tau s}\, \mathbf{u}_\tau^k = \mathbf{P}_{u\tau}^k \tag{45.5}$$

where T indicates the transpose and k the layer. $\mathbf{K}_{uu}^{k\tau s}$ and $\mathbf{P}_{u\tau}^k$ are the fundamental nuclei for the stiffness and load terms, respectively, and they are assembled through the depicted indexes, τ and s, which consider the order of the expansion in z for the displacements.

The corresponding Neumann-type boundary conditions are:

$$\boldsymbol{\Pi}_d^{k\tau s}\, \mathbf{u}_\tau^k = \boldsymbol{\Pi}_d^{k\tau s}\, \bar{\mathbf{u}}_\tau^k\,, \tag{45.6}$$

where $\boldsymbol{\Pi}_d^{k\tau s}$ is the fundamental nucleus for the boundary conditions and the over-line indicates an assigned condition.

For the explicit form of fundamental nuclei for the Navier-type closed-form solution and more details about the constitutive equations and geometrical relations for laminated plates and shells in the framework of CUF, one can refer to [6].

45.3 Method to Build the Best Plate/Shell Theories

Significant advantages are offered by refined plate/shell theories in terms of accuracy of the solution, but a higher computational effort is necessary because of the presence of a larger number of displacement variables. This work is an effort to understand the convenience of using a fully refined model rather than a reduced one. The effectiveness of each term, as well as the terms that have to be retained in the formulation, are investigated as follows.

1. The problem data are fixed (geometry, boundary conditions, loadings, materials and layer lay-outs).
2. A set of output variables is chosen (maximum displacements, stress/displacement component at a given point, etc.).
3. A theory is fixed, that is, the terms that have to be considered in the expansion of u, v, and w are established.
4. A reference solution is used to establish the accuracy (the $N = 4$ case is assumed as the best-reference solution since it offers an excellent agreement with the 3D solutions).
5. The effectiveness of each term is numerically established measuring the error produced compared to the reference solution.
6. Any term which does not give any contribution to the mechanical response is not considered as effective in the kinematics model.
7. The most suitable kinematics model is then detected for a given structural lay-out.

A graphical notation has been introduced to make the representation of the obtained results more readable. This consists of a table with three lines for the displacement components and a number of columns that depends on the number of displacement variables which are used in the expansion. All 15 terms of the expansion are reported in Table 45.1. The table is referred to the fourth-order model, $N = 4$, expressed in

Table 45.1 Locations of the displacement variables within the tables layout

$N=0$	$N=1$	$N=2$	$N=3$	$N=4$
u_1	$u_2\ z$	$u_3\ z^2$	$u_4\ z^3$	$u_5\ z^4$
v_1	$v_2\ z$	$v_3\ z^2$	$v_4\ z^3$	$v_5\ z^4$
w_1	$w_2\ z$	$w_3\ z^2$	$w_4\ z^3$	$w_5\ z^4$

Eq.(45.3). White and black triangles are used to denote the inactive and active terms respectively, as in Table 45.2. Table 45.3 shows the case in which the parabolic term of the expansion of the in-plane displacement v is discarded. The elimination of a

Table 45.2 Symbols to indicate the status of a displacement variable

Active term	Inactive term
▲	△

term, as well as the evaluation of its effectiveness in the analysis, can be obtained by exploiting a penalty technique. The corresponding results are compared with those given by a full fourth-order model using the percentage variations δ_w and δ_σ. These parameters are defined according to the following formulas:

Table 45.3 Symbolic representation of the reduced kinematics model with v_3 deactivated

▲	▲	▲	▲	▲
▲	▲	△	▲	▲
▲	▲	▲	▲	▲

$$\delta_w = \frac{w}{w_{N=4}} \times 100,\ \delta_{\sigma_{zz}} = \frac{\sigma_{zz}}{\sigma_{zz_{N=4}}} \times 100, \tag{45.7}$$

Where subscript '$N = 4$' denotes the values that correspond to the plate/shell theory given by Eq.(45.3). Parameters related to other stress or displacement values could also be introduced (δ_u, $\delta_{\sigma_{xx}}$, etc.).

It is important to notice that a displacement variable of the expansion can be considered non effective with respect to a specific output component (displacement or stress) when, if neglected (removed from the formulation), it does not introduce any changes in the results according to a fixed accuracy. The accuracy is here fixed to be as 0.05 %, that is, a term is considered negligible if the error caused by its absence in the kinematics model is lower than 0.05 %. When conducting the analysis of each displacement variable, a reduced kinematics model, if any exists, is established which is equivalent to a fourth-order expansion.

The numerical investigation has considered either plates and shells with different lay-outs: isotropic, orthotropic and cross-ply composite plates and shells. Furthermore, the effects on the definition of the reduced model of the following geometrical/mechanical parameters have been evaluated: length-to-thickness ratio a/h (for shells R/h), orthotropic ratio E_L/E_T and ply sequence.

45.4 Results and Discussion

In the following discussion, either plate and shell structures with different geometries and lay-outs are considered. For each case, it is taken for granted that the solution obtained with the theory of Eq.(45.3) is very close to the 3D solution, as demonstrated in many first-author's works among which [5,7,12,13,15]. Therefore, the fourth-order model has been chosen as the reference solution for the present analysis.

45.4.1 Plates

A simply supported plate has been considered. A bi-sinusoidal transverse distributed load is applied to the top surface:

$$p_z = \bar{p}_z\ \sin(\frac{m\pi x}{a})\ \sin(\frac{n\pi y}{b}) \tag{45.8}$$

with $a = 0.1$ [m], b is assumed equal to a. $\bar{p}_z$ is the applied load amplitude, $\bar{p}_z = 1$ [kPa], and m, n are the wave number in the two in-plane, plate directions. Attention has been restricted to the case $m = n = 1$. w, σ_{xx} and σ_{zz} are computed at $[a/2, b/2, h/2]$, while σ_{xz} is computed at $[0, b/2, 0]$. An isotropic plate has been considered first. Young's modulus, E, is equal to 73 [GPa]. Poisson's ratio, ν, is equal to 0.34. Four different length-to-thickness ratios, a/h, are considered: 100, 10, 5 and 2, that is, thin, moderately thick, thick and very thick plates, respectively.
The results of the effectiveness of each displacement variable are given in Table 45.4. A thin plate geometry has been considered ($a/h = 100$).

The percentage variation, δ, introduced neglecting each displacement variable, is evaluated for w, σ_{xx}, σ_{xz} and σ_{zz}. The following comments can be made.

1. The constant term w_1, the linear terms, u_2 and v_2 and the parabolic term w_3 are the most important ones to detect w, σ_{xx} and σ_{xz}.
2. Accurate evaluations of σ_{zz} require the use of $w_1,...,w_5$ variables.

For the sake of brevity, the tables referring to different a/h values are not reported here, but they can be found in [15]. A rather quite comprehensive analysis is instead given in Table 45.5, which considers different plate geometries. The sets of effective terms are reported, that is, the plate models required to detect the fourth-order solution are shown.

The last column gives the expansion terms needed to 'exactly' detect the whole considered outputs 'exactly'. M_e states the number of terms (i. e. computational costs) of the theory necessary to meet the fourth-order accuracy requirements. The required terms are again those corresponding to the black triangles. Some remarks can be made.

1. As a/h decreases, the theories become more computationally expensive (M_e increases).
2. Different choices of displacement variables are required to obtain exact different outputs.
3. All 15 terms are necessary for very thick plate geometries.

Table 45.5 is, as in the paper title, an attempt to offer both guidelines and recommendations for building the best plate/shell theories for the considered problems.

Orthotropic plates have been considered to assess the accuracy of the plate theory vs. orthotropic ratio, E_L/E_T. It is a well known fact that orthotropic plates, such as laminated composite structures, exhibit larger shear deformations than metallic structures made of isotropic materials. The analysis of such plates is therefore of particular interest for the present investigation.

Young's modulus along the transverse direction, E_T, is assumed as high as 1 [GPa]. Different orthotropic ratios, E_L/E_T, are assumed: 5, 25 and 100, where E_L stands for Young's modulus along the longitudinal direction. Young's thickness modulus, E_z, is assumed equal to E_T. The shear moduli are assumed as high as 0.39 [GPa]. Poisson's ratios, ν_{LT} and ν_{Lz}, are equal to 0.25, and a/h is assumed equal to 10.

The sets of effective variables for different orthotropic ratios are summarized in Table 45.6.

Table 45.4 Influence of each displacement variable of a fourth order model on the solution. $a/h=$ 100. Isotropic plate [15]

	δ_w [%]	$\delta_{\sigma_{xx}}$ [%]	$\delta_{\sigma_{xz}}$ [%]	$\delta_{\sigma_{zz}}$ [%]
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
△▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ △▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ △▲▲▲▲	1.3×10^{-5}	1.2×10^{-2}	1.3×10^{-2}	81.3
▲△▲▲▲ ▲▲▲▲▲ ▲▲▲▲▲	0.2	0.2	299.7	100.0
▲▲▲▲▲ ▲△▲▲▲ ▲▲▲▲▲	0.2	0.2	0.2	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲△▲▲▲	100.0	100.0	100.0	139.1
▲▲△▲▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲△▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲△▲▲	94.6	74.2	101.8	-8.1×10^{4}
▲▲▲△▲ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	72.3	100.0
▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲△▲	100.0	100.0	100.0	100.0
▲▲▲▲△ ▲▲▲▲▲ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲△ ▲▲▲▲▲	100.0	100.0	100.0	100.0
▲▲▲▲▲ ▲▲▲▲▲ ▲▲▲▲△	100.0	100.0	100.0	127.6

Table 45.5 Comparison of the sets of effective terms for isotropic plates with different a/h [15]

w	σ_{xx}	σ_{xz}	σ_{zz}	COMBINED
		$a/h = 100$		
$M_e = 4$	$M_e = 4$	$M_e = 5$	$M_e = 4$	$M_e = 7$
△▲△△△ △▲△△△ ▲△▲△△	△▲△△△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△▲△△	△△△△△ △△△△△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲▲▲△▲
		$a/h = 10$		
$M_e = 6$	$M_e = 10$	$M_e = 6$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲△▲△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△▲△ ▲△▲△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ △▲▲▲△ ▲▲▲▲▲
		$a/h = 5$		
$M_e = 9$	$M_e = 11$	$M_e = 7$	$M_e = 9$	$M_e = 13$
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		$a/h = 2$		
$M_e = 13$	$M_e = 14$	$M_e = 7$	$M_e = 13$	$M_e = 15$
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Table 45.6 Comparison of the sets of effective terms for orthotropic plates with different E_L/E_T [15]

w	σ_{xx}	σ_{xz}	σ_{zz}	COMBINED
		$E_L/E_T = 5$		
$M_e = 7$	$M_e = 10$	$M_e = 6$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲▲▲△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△▲△ ▲△▲△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ ▲▲▲▲△ ▲▲▲▲▲
		$E_L/E_T = 25$		
$M_e = 6$	$M_e = 10$	$M_e = 4$	$M_e = 9$	$M_e = 13$
△▲△▲△ △▲△▲△ ▲▲△△△	▲▲△▲△ ▲▲△▲△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲△△△△	▲△▲△△ ▲△▲△△ ▲▲▲▲▲	▲▲▲▲△ ▲▲▲▲△ ▲▲▲▲▲
		$E_L/E_T = 100$		
$M_e = 5$	$M_e = 6$	$M_e = 3$	$M_e = 7$	$M_e = 11$
△▲△▲△ △▲△△△ ▲▲△△△	▲▲△△△ ▲▲△△△ ▲▲△△△	△▲△△△ △▲△△△ ▲△△△△	△△△△△ ▲△▲△△ ▲▲▲▲▲	▲▲△▲△ ▲▲▲△△ ▲▲▲▲▲

The obtained results suggest the following comments.

1. The conclusions made for the isotropic plate are confirmed for the orthotropic plate.
2. The plate theory needed to furnish an accurate description of several outputs tends to have lower M_e for larger E_L/E_T values (as shown in the last column of Table 45.6);
3. The orthotropic ratio, E_L/E_T, plays a similar role as the length-to-thickness ratio, a/h; these two parameters are the most significant in evaluating the accuracy of a given plate theory.

Composite plates have been analyzed to assess the plate theory accuracy vs. stacking sequence. Attention is restricted to higher order theories, as in Eq. (45.3). The authors are aware that laminated structures require more adequate descriptions, such as those given by the so-called zig-zag theories as well as a layer-wise description. Such analysis are herein omitted but could be the subject of future investigations. The readers are addressed to the already mentioned review papers as well as to the historical review on zig-zag theories in [11].

A three-layer composite plate has been analyzed. E_L is equal to 40 [GPa]. E_T and E_z are equal to 1 [GPa]. ν_{LT} and ν_{Lz} are equal to 0.5 and 0.6, respectively. Each layer is 0.001 [m] thick. Three stacking sequences are considered: two symmetrical (0°) and (0°/90°/0°), and one asymmetrical (0°/0°/90°).

Table 45.7 shows the plate model for each stacking sequence and output variable as well as for the combined evaluation of w, σ_{xx}, σ_{xz} and σ_{zz} to obtain a fourth-order model accuracy.

Table 45.7 Comparison of the sets of effective terms for composite plates with different stacking sequences [15]

w	σ_{xx}	σ_{xz}	σ_{zz}	COMBINED
		0°		
$M_e = 5$	$M_e = 5$	$M_e = 4$	$M_e = 4$	$M_e = 7$
△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△△△△	△△△△△ △△△△△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲▲▲△▲
		0°/90°/0°		
$M_e = 5$	$M_e = 5$	$M_e = 4$	$M_e = 7$	$M_e = 7$
△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△▲△△	△▲△▲△ △▲△△△ ▲△△△△	△▲△▲△ △▲△△△ ▲▲▲△▲	△▲△▲△ △▲△△△ ▲▲▲△▲
		0°/0°/90°		
$M_e = 9$	$M_e = 12$	$M_e = 10$	$M_e = 14$	$M_e = 14$
▲▲▲▲▲ ▲▲△△△ ▲△▲△△	▲▲▲▲▲ ▲▲△△△ ▲▲▲▲▲	▲▲▲▲▲ ▲▲△△△ ▲▲▲△△	▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲	▲▲▲▲▲ ▲▲▲△▲ ▲▲▲▲▲

The table highlights the following main aspects related to the choice of the plate theory.

1. The stacking sequence influences the construction of adequate plate models to a great extent; it plays a similar role as the geometry and the orthotropic ratio;
2. An asymmetric lamination sequence requires a considerably higher number of displacement variables than a symmetric one.

45.4.2 Shells

An isotropic shell has been considered first. The geometry of the shell is described in Fig. 45.1. Referring to Ren's work [39], a simply supported shell and a sinusoidal

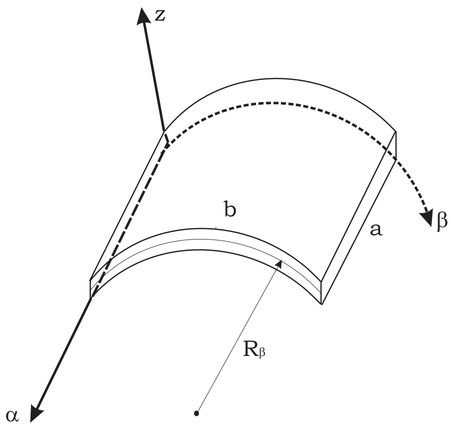

Fig. 45.1 Geometry of shell

distribution of transverse pressure applied at the top surface have been considered (cylindrical bending problem):

$$p_z = \bar{p}_z \sin\left(\frac{n\pi\beta}{b}\right) \tag{45.9}$$

where β is the curvilinear coordinate. The attention has been restricted to the case $n = 1$. R_β is assumed equal to 10 [m] and the dimension $b = \frac{\pi}{3}R_\beta$. The amplitude of the applied load is $\bar{p}_z = 1$ [kPa]. Young's modulus, E, is equal to 73 [GPa] and Poisson's ratio, ν, is equal to 0.34. Four different thickness ratios, R_β/h, are considered: 100, 50, 10 and 4. The displacement w and the stresses σ_{yy} and σ_{zz} are computed at $[a/2, b/2, h/2]$, while σ_{yz} is computed at $[a/2, 0, 0]$.

For the sake of brevity, the study of the effectiveness of each displacement variable is not here reported, but in Table 45.8 the sets of effective terms required to

Table 45.8 Comparison of the sets of effective terms for isotropic shells with different R_β/h

w	σ_{yy}	σ_{yz}	σ_{zz}	COMBINED
		$R_\beta/h = 100$		
$M_e = 4$	$M_e = 6$	$M_e = 5$	$M_e = 9$	$M_e = 9$
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		$R_\beta/h = 50$		
$M_e = 4$	$M_e = 6$	$M_e = 5$	$M_e = 10$	$M_e = 10$
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		$R_\beta/h = 10$		
$M_e = 6$	$M_e = 8$	$M_e = 5$	$M_e = 10$	$M_e = 10$
△△△△△ ▲▲△▲△ ▲△▲▲△	△△△△△ ▲▲△▲△ ▲▲▲▲▲	△△△△△ ▲▲△▲△ ▲△▲△△	△△△△△ ▲▲▲▲▲ ▲▲▲▲▲	△△△△△ ▲▲▲▲▲ ▲▲▲▲▲
		$R_\beta/h = 4$		
$M_e = 6$	$M_e = 9$	$M_e = 8$	$M_e = 10$	$M_e = 10$
△△△△△ ▲▲△▲△ ▲△▲▲△	△△△△△ ▲▲△▲△ ▲▲▲▲▲	△△△△△ ▲▲△▲▲ ▲△▲▲▲	△△△△△ ▲▲▲▲▲ ▲▲▲▲▲	△△△△△ ▲▲▲▲▲ ▲▲▲▲▲

detect the fourth-order solution are reported for each thickness ratio R_β/h. The conclusions made for the isotropic plate are valid also for the shell: as the thickness ratio decreases, the theories become more computationally expensive (M_e increases). Moreover, as in the plate w_3 is important to detect w, σ_{yy} and σ_{yz} and all the terms of w expansion are necessary for the exact evaluation of σ_{zz}. In this particular case, the terms of the expansion of u are non influential because a cylindrical bending problem has been considered. One can note that the constant term of the in plane displacement v is more important in the shell than in the plate because the shell has a membranal deformation even when it is very thin. This fact is due to the curvature.

Orthotropic shells have been considered to assess the accuracy of the shell theory vs. orthotropic ratio, E_L/E_T. The geometry of the shell is cylindrical, with $a/R_\beta = 4$, and the loading is internal sinusoidal pressure $p_z = \bar{p}_z\, sin(\frac{m\pi\alpha}{a})\, sin(\frac{n\pi\beta}{b})$, with $m = 1$ and $n = 8$. Young's modulus along the transverse direction, E_T, is assumed as high as 1 [GPa]. Different orthotropic ratios, E_L/E_T, are assumed: 5, 25 and 100, where E_L stands for Young's modulus along the longitudinal direction. The shear moduli G_{LT} and G_{TT} are assumed $0.5E_T$ and $0.2E_T$, respectively, and Poisson's ratios, ν_{LT} and ν_{TT}, are equal to 0.25. R_β/h is assumed equal to 100. The displacement w and the stresses σ_{yy} and σ_{zz} are computed at $[a/2, b/2, -h/2]$, while σ_{yz} is computed at $[a/2, 0, 0]$. The 3D solution for this problem is given by Varadan and Bhaskar in [41].

A single-layered (90°) shell has been analyzed (the lamination angle is measured with respect to the longitudinal axis). The sets of effective variables for different orthotropic ratios are summarized in Table 45.9.

Table 45.9 Comparison of the sets of effective terms for orthotropic shells with different E_L/E_T

w	σ_{yy}	σ_{yz}	σ_{zz}	COMBINED
		$E_L/E_T = 5$		
$M_e = 7$	$M_e = 8$	$M_e = 7$	$M_e = 10$	$M_e = 11$
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		$E_L/E_T = 25$		
$M_e = 8$	$M_e = 8$	$M_e = 7$	$M_e = 10$	$M_e = 10$
▲▲△△△ ▲▲△▲△ ▲▲▲△△	▲▲△△△ ▲▲△▲△ ▲▲▲△△	▲▲△△△ ▲▲△▲△ ▲▲△△△	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲
		$E_L/E_T = 100$		
$M_e = 7$	$M_e = 7$	$M_e = 7$	$M_e = 10$	$M_e = 11$
▲▲△△△ ▲▲△▲△ ▲▲△△△	▲▲△△△ ▲▲△▲△ ▲△▲△△	▲▲△△△ ▲▲△▲△ ▲▲△△△	△▲△△△ ▲▲△▲▲ ▲▲▲▲▲	▲▲△△△ ▲▲△▲▲ ▲▲▲▲▲

The obtained results confirm the conclusions made for the plate. The orthotropic ratio plays a similar role as the thickness ratio in evaluating the accuracy of a given theory and the sets of effective terms differ to a great extent with changing of E_L/E_T.

Finally, composite shells have been analyzed to assess the shell theory accuracy vs. stacking sequence. In addition to the shell considered above, a three-layered symmetrical (90°/0°/90°) shell and a two-layered asymmetric (90°/0°) shell have been considered. In all the cases, the layers are of equal thickness. The orthotropic ratio E_L/E_T is taken equal to 25 and the thickness ratio $R_\beta/h = 100$.
Table 45.10 shows the shell model for each stacking sequence in order to obtain a fourth-order model accuracy. As in the plate case, the table highlights that the stacking sequence influences the construction of adequate models to a great extent and an asymmetric lamination sequence requires a higher number of displacement variables than a symmetric one.

In general, from the analysis of the shell geometry, one can note that more and different displacement variables are effective in the evaluation of the different outputs, compared with the plate, if analogous problems are considered. This fact is due to the curvature that appear in the strain-displacement relations and couples membranal and bending behaviors of the shell.

Table 45.10 Comparison of the sets of effective terms for composite shells with different stacking sequences

w	σ_{yy}	σ_{yz}	σ_{zz}	COMBINED
		90°		
$M_e = 8$	$M_e = 8$	$M_e = 7$	$M_e = 10$	$M_e = 10$
▲▲△△△ ▲▲△▲△ ▲▲▲△△	▲▲△△△ ▲▲△▲△ ▲▲▲△△	▲▲△△△ ▲▲△▲△ ▲▲△△△	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲
		90°/0°/90°		
$M_e = 8$	$M_e = 9$	$M_e = 6$	$M_e = 10$	$M_e = 10$
▲▲△△△ ▲▲△▲△ ▲▲▲△△	▲▲△△△ ▲▲△▲△ ▲▲▲▲△	▲▲△△△ ▲▲△▲△ ▲△△△△	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲	▲▲△△△ ▲▲△▲△ ▲▲▲▲▲
		90°/0°		
$M_e = 6$	$M_e = 9$	$M_e = 10$	$M_e = 11$	$M_e = 12$
▲▲△△△ ▲▲△△△ ▲△▲△△	▲▲△△△ ▲▲△△△ ▲▲▲▲▲	▲▲△△△ ▲▲▲▲▲ ▲▲▲△△	▲▲△△△ ▲▲▲△▲ ▲▲▲▲▲	▲▲△△△ ▲▲▲▲▲ ▲▲▲▲▲

45.4.3 The Best Theory Diagram

The approach here presented has proved its validity in constructing:

1. reduced models equivalent to a full higher-order theory;
2. reduced models able to fulfil a given accuracy input.

The construction of these models has highlighted that Unified Formulation allows us, for a given problem, to obtain a diagram that in terms of accuracy (input) gives an answer to the following fundamental questions:

- what is the 'minimum' number of the terms, N_{min}, to be used in a plate/shell model?
- Which are the terms to be retained, that is, which are the generalized displacement variables to be used as degrees od freedom?

To the best of the authors' knowledge, there are no other available methods that can provide this kind of results. The present method of analysis is able to create plots like the one in Fig. 45.2 that gives the number of terms as function of the permitted error.

This plot can be defined as the Best Theory Diagram BTD since it allows us to edit an arbitrary given theory in order to have a lower amount of terms for a given error (vertical shift, Δ_N) or, to increase the accuracy keeping the computational cost constant (horizontal shift, Δ_{error}). Most times, the plot presented appears as an hyperbole. CUF makes the computation of such a plot possible. Note that the diagram has the following properties:

- it changes by changing problems (thickness ratio, orthotropic ratio, stacking sequence, etc.);

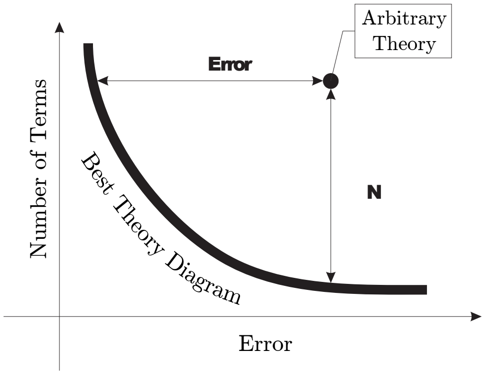

Fig. 45.2 An example of Best Theory Diagram (BTD) [16]

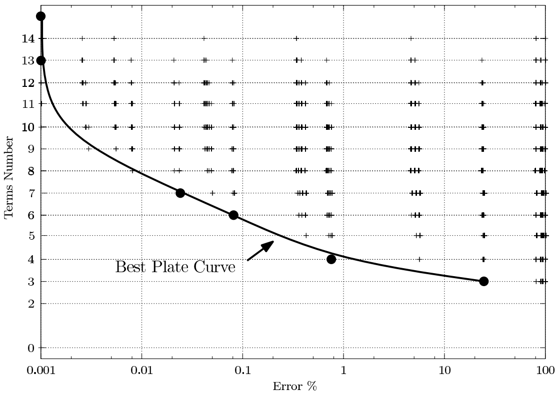

Fig. 45.3 Accuracy of all the possible combinations of plate models in computing w for the simply-supported plate loaded by a bi-sinusoidal load (each '+' indicates a different plate model) [16]

- it changes by changing output variable (displacement/stress components, or a combination of these).

The validity of the BTD is tested by computing the accuracy of all the plate/shell models obtainable as a combination of the 15 terms of the fourth-order theory. The results are reported in Fig. 45.3 in the case of a simply-supported plate loaded by a bi-sinusoidal load; the transversal displacement w is considered as output variable.

The BTD perfectly matches the lower boundaries of the region where all the models lie. This confirms that the BTD represents the best theory (i.e. the least cumbersome) for a given problem. The BTD permits the evaluation of any existing plate/shell model, as in the previous sections. The distance from the BTD of a given known model represents a guideline to recommend any other theory. More complex analyses will be conducted using the Finite Element Method in future work to investigate the effects of loadings, boundary conditions, etc. Some of these analyses have been already studied in [14, 16] where the attention was restricted to plates. Future investigations could consider the shells.

45.5 Conclusions

The effectiveness of each displacement variable of higher order plate/shell theories has been investigated in this paper. The Carrera Unified Formulation (CUF) has been used for the systematic implementation of refined models. Navier-type closed-form solutions have been adopted for the analysis. Isotropic, orthotropic and composite plates and shells have been considered. The role of each displacement variable has been described in terms of displacement and stress components, referring to a fourth-order model solution. The contribution of each term to the accuracy of the solution has been evaluated, introducing the so-called mixed axiomatic/asymptotic method, which is able to recognize the effectiveness of each displacement variable of an arbitrary refined theory.

It can be stated that the choice of the model which suits the accuracy requirements for a given problem is dominated by the length-to-thickness ratio, the orthotropic ratio and the lamination sequence. It has also been found that each displacement/stress component would require its own model to obtain exact results. Moreover, the number of retained terms is very closely related to the geometrical/mechanical configuration of the considered problem. In particular, the shell configurations require more displacement variables than the plates because the curvature introduces coupling effects. Remarkable benefits, in terms of total amount of problem variables, are obtained for thin structures or for symmetrical laminations. Finally, the use of full models is mandatory when a complete set of results is needed.

CUF has shown to be well able to deal with a method that could be stated as a *mixed assiomatic/asymptotic* structural analysis of different structures. Two main benefits can be obtained.

1. It permits the accuracy of each problem variable to be evaluated by comparing the results with more detailed analyses (also provided by CUF); no mathematical/variational techniques are needed as in the case of asymptotic-type analyses.
2. It offers the possibility of considering the accuracy of the results as an input, while the output is represented by the set of displacement variables which are able to fulfill the requirement.

From this analysis it is possible to draw a curve, the Best Theory Diagram BTD, that allows us to edit an arbitrary given theory in order to have a lower amount of terms for a given error or to increase the accuracy keeping the computational cost constant.

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